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THE EVALUATION OF BESSEL AND NEUMANN FUNCTIONS THROUGH THE USE --ETC(U)

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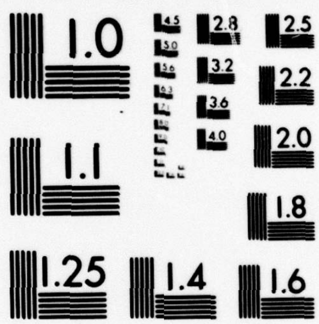

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TRACOR

TECHNICAL MEMORANDUM

THE EVALUATION OF BESSEL AND NEUMANN FUNCTIONS THROUGH THE USE OF TAYLOR SERIES SOLUTIONS TO BESSEL'S DIFFERENTIAL EQUATION

Prepared for

The Bureau of Ships
Code 689B

December 3, 1962

TRACOR, INC.

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MOST Project

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LEVEL II

TECHNICAL MEMORANDUM

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This technical memorandum contains partial results obtained during an analytical study of the sound field near a dome-baffle-transducer complex.

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THE EVALUATION OF BESSEL AND NEUMANN FUNCTIONS THROUGH THE USE OF TAYLOR SERIES SOLUTIONS TO BESSEL'S DIFFERENTIAL EQUATION

I. INTRODUCTION

The solutions of boundary value problems expressed in cylindrical coordinates, for example the sound field resulting from a plane wave incident on a baffle-transducer, are often expanded in series containing Bessel and Neumann functions. The ability to evaluate such solutions depends on the availability of numerical values for these functions. Tables of Bessel and Neumann functions are presently available. However, these tables are somewhat limited in regard to the range of order and argument, and it is not convenient, because of computer storage limitations, to store function tables in order to evaluate problems involving a large number of Bessel and Neumann functions ($J_n(z)$ and $N_n(z)$ respectively). Consequently, in the evaluation of problems requiring Bessel and Neumann functions having a large range of order and argument, an analytical procedure which can be used to evaluate the $J_n(z)$ and $N_n(z)$ for such a range of z and n is required.

The usual power series representations of $J_n(z)$ and $N_n(z)$ may be used only if z and n are small. If z is large, a limited number of successive terms in the power series may be monotonically increasing. In such cases, due to the limited numerical significance available in most computers, accurate results cannot be obtained. However, for large orders and arguments, approximate expressions exist for evaluating $J_n(z)$ and $N_n(z)$. Such expressions are found for example, in Watson.[1] Unfortunately, the regions in which the power series and the approximations yield accurate results do not include the entire range of order and argument required.

The purpose of this technical memorandum is to illustrate a procedure for calculating $J_{n}^{sub}(z)$ and $N_{n}^{sub}(z)$ which will extend the present computation capability to a somewhat larger range of n and z. This procedure is based on Taylor series solutions

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to Bessel's differential equation expanded about points near the argument desired for any particular $J_n(z)$ and $N_n(z)$.

While extending the computational capability presently available, this procedure does not permit calculations over the entire range of arguments and orders not covered by the power series or asymptotic expansions. However, an improvement on this procedure is currently being developed and will be described in a subsequent technical memorandum. This further development promises a capability for computing $J_n(z)$ and $N_n(z)$ for a continuous range of orders and arguments, including those values of n and z not covered by the previously stated techniques.

II. TAYLOR SERIES SOLUTIONS TO BESSEL'S EQUATION

A. The Method

The method used in obtaining solutions to Bessel's equation is based on the Taylor series expansion of the solution about a non-singular point. Details of this method are available in most texts on ordinary differential equations.[2]

Basically, the points to be noted are as follows. First, Bessel's equation is linear and homogeneous with only one singular point, that being a regular singular point at $z = 0$. Therefore, solutions to the equation have Taylor series expansions about any point $z = a$, where $a > 0$; and the series are convergent in the interval $|z - a| < a$. Second, since the equation is of second order and is linear, there exist exactly two linearly independent solutions. Therefore the new solutions will be linear combinations of $J_n(z)$ and $N_n(z)$.

B. The Taylor Series Solutions

Bessel's equation can be written in the form

$$z^2 \frac{d^2y}{dz^2} + z\frac{dy}{dz} + (z^2-n^2)y = 0. \tag{1}$$

Solutions to Taylor Series Solutions of Bessel's equation expanded about $z = a$ may be written as

$$y = \sum_{m=0}^{\infty} A_m(z - a)^m \qquad \text{for } a > 0. \tag{2}$$

The process of determining the A_m is facilitated by applying the transformation

$$x = z - a \tag{3}$$

to the expressions (1) and (2).

Bessel's equation then becomes

$$(x^2+2ax+a^2)\frac{d^2y}{dx^2} + (x+a)\frac{dy}{dx} + (x^2+2ax+a^2-n^2)\,y = 0, \tag{4}$$

and the solution becomes

$$y = \sum_{m=0}^{\infty} A_m\, x^m. \tag{5}$$

If this series is substituted into Equation (4) the result is:

$$\left\{2a^2A_2+aA_1+(a^2-n^2)A_o\right\} + \left\{6a^2A_3+6aA_2+(1+a^2-n^2)A_1+2aA_o\right\} x$$
$$+\sum_{m=2}^{\infty}\Big\{(m+2)\ (m+1)a^2A_{m+2}+(2m+1)\ (m+1)aA_{m+1}+(m^2+a^2-n^2)A_m$$
$$+ 2aA_{m-1}+A_{m-2}\Big\}\ x^m = 0. \tag{6}$$

Equation (6) is an identity in x. Hence, the coefficient of each power of x must vanish. Thus

$$A_2 = -\frac{1}{2a^2}\left[aA_1 + (a^2 - n^2)A_o\right] \quad , \tag{7a}$$

$$A_3 = -\frac{1}{6a^2}\left[6aA_2 + (1 + a^2 - n^2)A_1 + 2aA_o\right] \quad , \tag{7b}$$

and, for $m \geq 2$,

$$A_{m+2} = -\frac{1}{(m+2)\ (m+1)a^2}\Big[(2m+1)\ (m+1)aA_{m+1} + (m^2+a^2-n^2)A_m$$
$$+ 2aA_{m-1} + A_{m-2}\Big]. \tag{7c}$$

A more convenient form of equation (7c) is

$$A_{m+4} = -\frac{1}{(m+4)\ (m+3)a^2}\Big\{(2m+5)\ (m+3)aA_{m+3} + \left[(m+2)^2+a^2-n^2\right]A_{m+2}$$
$$+ 2aA_{m+1} + A_m\Big\}\ . \tag{7d}$$

Equations (7) define A_2, A_3, A_4 ... in terms of A_o and A_1. It is therefore possible to define two independent solutions to Equation (4) by an arbitrary choice of the A_o and A_1. One such solution is obtained by setting $A_o = 1$ and $A_1 = 0$. This solution is denoted by $P_n(z)$ and is given by

$$P_n(z) = \sum_{m=0}^{\infty} B_m \quad (z - a)^m. \tag{8}$$

The B_m are

$$B_m = A_m \text{ for } m = 0,1,2,3,4. \; . \; . \; . \tag{9}$$

Thus

$$P_n(z) = 1 - [\frac{(a^2-n^2)}{2a^2}] \, (z-a)^2 + [\frac{(a^2-3n^2)}{6a^3}] \, (z-a)^3$$

$$+ \sum_{m=0}^{\infty} B_{m+4} \quad (z-a)^{m+4}, \tag{10}$$

where

$$B_o = 1 \, , \tag{11a}$$

$$B_1 = 0 \, , \tag{11b}$$

$$B_2 = - \frac{(a^2-n^2)}{2a^2} \quad , \tag{11c}$$

$$B_3 = \frac{a^2-3n^2}{6a^3} \quad , \tag{11d}$$

and for $m \geq 0$

$$B_{m+4}, = - \frac{1}{(m+4) \; (m+3)a^2} \left\{ (2m+5) \; (m+3)a \; B_{m+3}, \right.$$

$$\left. + [(m+2)^2 + a^2-n^2] \; B_{m+2}, + 2aB_{m+1}, + B_m, \right\} \quad . \tag{11e}$$

A second solution is obtained by taking $A_o = 0$ and $A_1 = 1$. This solution is denoted by $Q_n(z)$ and is given by

$$Q_n(z) = \sum_{m=0}^{\infty} C_m (z-a)^m . \qquad (12)$$

As before

$$C_m = A_m \text{ for } m = 0,1,2,3,4. \; . \; . \; .$$

Thus

$$Q_n(z) = (z-a) - (\frac{1}{2a}) (z-a)^2 + [\frac{2 - (a^2-n^2)}{6a^2}] (z-a)^3$$

$$+ \sum_{m=0}^{\infty} C_{m+4} (z-a)^{m+4} \qquad (13)$$

where

$$C_o = 0, \qquad (14a)$$

$$C_1 = 1, \qquad (14b)$$

$$C_2 = - \frac{1}{2a} \qquad (14c)$$

$$C_3 = [\frac{2 - (a^2-n^2)}{6a^2}] \qquad (14d)$$

and for $m \geq 0$

$$C_{m+4} = - \frac{1}{(m+4) (m+3)a^2} \left\{ (2m+5) (m+3)a C_{m+3}, \right.$$

$$\left. + [(m+2)^2 + a^2-n^2] C_{m+2} + 2a C_{m+1} + C_m \right\} . \qquad (14e)$$

The general solution of Bessel's equation is thus

$$y = F P_n(z) + G Q_n(z) \qquad (15).$$

where F and G are arbitrary constants.

C. Convergence of the Series

It is a well known result of differential equation theory that Taylor series solutions are uniformly convergent in the interval $0 < z < 2a$, where a singularity of the differential equation exists at $z = 0$, a is the point of expansion, and no singularities exist in the interval $0 < z < 2a$. Hence, convergence of the method is assured. What is important from the standpoint of computation is the rate of convergence of the series. It has previously been suggested that for an accurate evaluation of the series, the magnitude of the successive terms of the series should be monotonically non-increasing, that is, the sequences $|B_m(z-a)^m|$ and $|C_m(z-a)^m|$ (see Equations 8 and 12) should both be monotonically non-increasing. Equations 11e and 14e indicate that the above terms are functions of both n and a. Hence the desired behavior of these terms is not assured merely by choosing $|z-a|$ small. The value of n must also be considered. At the present time no rigorous criterion has been developed for choosing a as a function of n in order to assure the desirable convergence behavior.

III. CALCULATION OF $J_n(z)$ AND $N_n(z)$

The solutions to Bessel's differential equation described in the previous section can be used to compute the familiar solutions $J_n(z)$ and $N_n(z)$. Specifically, the procedure of computing the values of $J_n(z)$ will be shown. This will be adequate since the procedure for obtaining the $N_n(z)$ parallels the procedure for $J_n(z)$.

The series expansion for $J_n(z)$ converges very rapidly for small values of z. As z takes on larger values convergence becomes slower. Quantitatively, the maximum value of z for which machine computation by use of the series is reasonable is that value for which the magnitude of successive terms in the series are monotonically non-increasing. The magnitude of the mth term in the series for $J_n(z)$ is

$\frac{(z/2)^{n + 2m}}{m!\ (m + n)!}$, and the magnitude of next term is

$\frac{(\frac{z}{2})^{n + 2m + 2}}{(m+1)!\ (m+n+1)!}$. The ratio of the second term to the previous one is

$\frac{(z/2)^2}{(m + 1)\ (m + n + 1)}$. The largest value of the ratio for any given z and n occurs when m = o, i.e.,

$\frac{(z/2)^2}{(n + 1)}$. For a monotonically non-increasing series this ratio must not be greater than unity, so that

$$z \leq 2\ \sqrt{n + 1}\ . \tag{16}$$

Although the series for $J_n'(z)$ is not monotonically non-increasing for this range of z, it is nearly enough so as to allow its use in these computations. If a $J_n(z)$ is needed for arguments larger than

those shown in Equation 16, the series approach will not yield useful results. The values for $J_n(z)$ for $z > 2\sqrt{n+1}$ can be determined using the P_n and Q_n functions.

The P_n and Q_n functions have been shown to be linearly independent solutions to Bessel's equation, hence the Bessel function can be written as a linear combination of P_n and Q_n

$$J_n(z) = A\, P_n(z) + B\, Q_n(z)\ . \tag{17}$$

From the definition of $P_n(z)$ and $Q_n(z)$ it is seen that

$$P_n(a) = 1, \tag{18a}$$

$$P_n'(a) = 0, \tag{18b}$$

$$Q_n(a) = 0, \tag{18c}$$

and

$$Q_n'(a) = 1. \tag{18d}$$

Evaluating $J_n(a)$ in Equation (17) and noting Equation (18) it is seen that

$$A = J_n(a). \tag{19a}$$

The value of $J_n'(a)$ from Equation (17) and Equation (18) shows that

$$B = J_n'(a). \tag{19b}$$

Equation (17) therefore may be written

$$J_n(z) = J_n(a)\, P_n(z) + J_n'(a)\, Q_n(z). \tag{20}$$

The values of $J_n(a)$ and $J_n'(a)$ in Equation (20) can be obtained by use of the usual series up to $a=2\sqrt{n+1}$. Substitution of this value of a into Equation (20) yields

$$J_n(z) = J_n(2\sqrt{n+1})\, P_n(z) + J_n'(2\sqrt{n+1})\, Q_n(z). \tag{21}$$

There exists a neighborhood about $z = 2\sqrt{n+1}$ in which the series for $P_n(z)$, $Q_n(z)$ and their derivatives will be monotonically non-increasing. The values of $J_n(z)$ in the neighborhood of $2\sqrt{n+1} < z \leq 2\sqrt{n+1} + \epsilon_1$ can be obtained through use of Equation (21).

It should be noted that ϵ_1 must satisfy the relation

$$\epsilon_1 < 2\sqrt{n+1} \tag{22}$$

since the radius of convergence of the series is $2\sqrt{n+1}$. If $J_n(z)$ is needed for still a larger value of z than $2\sqrt{n+1} + \epsilon_1$, $J_n(2\sqrt{n+1} + \epsilon_1)$ and $J_n'(2\sqrt{n+1} + \epsilon_1)$ can be computed using Equation (21). Taking a in Equation (20) to be $(2\sqrt{n+1} + \epsilon_1)$, the result is

$$J_n(z) = J_n(2\sqrt{n+1} + \epsilon_1)\, P_n(z) + J_n'(2\sqrt{n+1} + \epsilon_1)\, Q_n(z) \tag{23}$$

There exists a neighborhood about $z = 2\sqrt{n+1} + \epsilon_1$ in which the series for $P_n(z)$, and $Q_n(z)$ will be monotonically non-increasing. Thus J_n can be evaluated for all z up to $2\sqrt{n+1} + \epsilon_1 + \epsilon_2$. Note that ϵ_2 must satisfy.

$$\epsilon_2 < 2\sqrt{n+1} + \epsilon_1. \tag{24}$$

If $J_n(z)$ for still a larger range of z is needed, this procedure may be repeated until the interval $2\sqrt{n+1} + \epsilon_1 + \epsilon_2 - - - \epsilon_p$ is large enough to cover the domain of interest.

This method of calculating J_n and N_n has been programmed and results have been obtained for $J_o(z)$, $J_1(z)$, $J_2(z)$, $N_o(z)$, $N_1(z)$, and $N_2(z)$ for $3 \leq z \leq 10$ and increments in z of $\Delta z = 0.5$. These results were compared with six-place tabulated values found in Reference (1). The calculated values agree to five significant figures.

REFERENCES

1. G. N. Watson, A Treatise on the Theory of Bessel Functions, The University Press, Cambridge, 1958.

2. Earl D. Rainville, Elementary Differential Equations, The Macmillan Company, New York, 1958.